

## Exercise 2, Set 12, Detailed answer

For the modeled process, we can see that  $\theta = 1$ ,  $\tau = 5$  and  $K = 2$ .

When fine-tuning a PID controller, we need to take measurements at time points that differ by very little relative to the timescale of the actual process. Since in this case the timescale is in the order of 5 seconds, we can select a time point difference of

$$\Delta T = T_m = 5/20 = 0.25$$

We can use the Ziegler–Nichols method to tune the parameters of the PID controller:

P controller:  $K_r = \frac{\tau}{\theta K} = \frac{5}{1 \cdot 2} = 2.5$

PI controller:  $K_r = 0.9 \frac{\tau}{\theta K} = 2.25$ ,  $t_I = 3.33\theta = 3.33$

PID controller:  $K_r = 1.2 \frac{\tau}{\theta K} = 3$ ,  $t_I = 2\theta = 2$ ,  $t_d = 0.5\theta = 0.5$

Now to find the numerical values for the control law we need to find the general formulas:

### P Controller

The proportional control law is:

$$N(t) = K_r e(t)$$

We can define two measurement points as:

$$e_k = e(t_k), \quad e_{k-1} = e(t_{k-1}), \quad \text{given } \Delta T = T_m = t_k - t_{k-1}$$

Then:

$$\Delta N_k = N(t_k) - N(t_{k-1}) = K_r (e_k - e_{k-1}) = \alpha e_k + \beta e_{k-1} + \gamma e_{k-2}$$

$$\alpha = K_r, \quad \beta = -K_r, \quad \gamma = 0$$

Numerical values:

$$\alpha = 2.5, \quad \beta = -2.5, \quad \gamma = 0$$

### PI Controller

PI control law:

$$N(t) = K_r \left( e(t) + \frac{1}{t_I} \int_0^t e(t') dt' \right)$$

Approximate the integral using small time steps:

$$\int_0^{t_k} e(t') dt' \approx T_m \sum_{i=0}^k e_i \quad \text{and} \quad N_k = K_r \left( e_k + \frac{T_m}{t_I} \sum_{i=0}^k e_i \right)$$

Thus:

$$\Delta N_k = K_r \left[ (e_k - e_{k-1}) + \frac{T_m}{t_I} e_k \right]$$

$$\alpha = K_r \left( 1 + \frac{T_m}{t_I} \right), \quad \beta = -K_r, \quad \gamma = 0$$

Numerical values:

$$\alpha \approx 2.42, \quad \beta = -2.25, \quad \gamma = 0$$

## PID Controller

PID control law:

$$N(t) = K_r \left( e(t) + \frac{1}{t_I} \int_0^t e(t') dt' + t_d \frac{de(t)}{dt} \right)$$

Approximate derivative:

$$\frac{de(t)}{dt} \approx \frac{e_k - e_{k-1}}{T_m} \text{ and } N_k = K_r \left( e_k + \frac{T_m}{t_I} \sum_{i=0}^k e_i + t_d \frac{e_k - e_{k-1}}{T_m} \right)$$

Thus:

$$\Delta N_k = K_r \left[ (e_k - e_{k-1}) + \frac{T_m}{t_I} e_k + \frac{t_d}{T_m} (e_k - 2e_{k-1} + e_{k-2}) \right]$$

$$\alpha = K_r \left( 1 + \frac{T_m}{t_I} + \frac{t_d}{T_m} \right), \quad \beta = K_r \left( -1 - 2 \frac{t_d}{T_m} \right), \quad \gamma = K_r \left( \frac{t_d}{T_m} \right)$$

Numerical values:

$$\alpha = 9.375, \quad \beta = -15, \quad \gamma = 6$$